SOLUTIONS OF SOME BOUNDARY VALUE PROBLEMS OF STATIC THEORY OF ELASTICITY FOR A LAYER WITH A SPHERICAL CAVITY

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In this paper, with the help of the method used in [1] for solution of the electrostatic (hydrodynamic) problem, some boundary value problems of the theory of elasticity are solved for the same regions, i.e. for the plane (parallel) layer with a spherical cavity. The method used in [1] can be described briefly in the following manner.

The solution, u, of the boundary value problem is sought in the form

$$u = u_0 + u_1 \tag{0.1}$$

where u_0 is the solution of the same boundary conditions on the planes, but without the cavity. It is easily found.

The additive term u_1 is sought in the form of a series of spherical functions relative to the center of the cavity, and to all its mirror images in the boundary planes of the layer.

The distribution of these points is symmetrical with respect to each plane. Because of this symmetry the homogeneous boundary conditions on the planes which should be satisfied by u_1 are satisfied by some simple relations between the coefficients of the series.

In order to satisfy the boundary conditions on the surface of the cavity, the expression u_1 is reduced to the single variables v, θ, and ϕ referred to the center of the cavity. To obtain this, the transformation formulas of the spherical functions for the translation of the origin are used [2].

From the boundary conditions on the surface of the cavity for the coefficients of the series, we obtain an infinite system of algebraic equations of the type

$$z_k + \sum_{l=1}^{\infty} c_{kl} z_l = b_k \qquad (k=1,2,3,\ldots) \tag{0.2}$$

where

$$\sum_{k=1}^{\infty} \sum_{l=1}^{\infty} c_{kl}^2 < \infty, \qquad \sum_{k=1}^{\infty} b_k^2 < \infty \tag{0.3}$$

The system (0.2) is regular for a large range of the values of the ratio of the radius of the cavity, R, to the thickness of the layer, a, (thus, in consequence of (0.3) it is completely regular*). The solution of this system is found by a reduction method [3], and also, at least in the region of its regularity, by the method of successive approximations.

In an analogous manner we can obtain solutions of the boundary value problems for a layer with an arbitrary namuber of spherical cavities whose centers lie on a straight line perpendicular to the faces of the layer.**

To use the above described method for the solutions of the boundary value problems in the theory of elasticity, it would be necessary to find first some complete system of solutions of the static equilibrium equation of the theory of elasticity

$$\triangle \mathbf{u} + \frac{1}{1-2\sigma} \operatorname{grad} \operatorname{div} \mathbf{u} = 0 \tag{0.4}$$

($\mathbf{u}$ is the displacement vector and σ is Poisson's ratio), which would correspond to the system of spherical functions in potential theory, and which would be suitable for application of the above method.*** Furthermore, the transformation formulas for these solutions should also be determined.

* Here the terminology of [3] is used.

** We note here, that the method of mirror images and use of the transformation formulas for spherical functions was employed by A.M. Rodov for solutions of some electrostatic problems of a half-space with a spherical cavity, (oral communication). For determination of the coefficients of the series Rodov obtained a system of the form (0.2).

***We could apply the system of solutions due to Thomson [4]. However, it was verified by the author that the systems obtained for determination of the coefficients were very unwieldy. Besides, for the Thomson's solutions great difficulties will be encountered in finding the so-called rotation formulas (see the end of Section 2).

1. System of normal solutions and determination of transformation formulas for this system.

(1) The above mentioned complete system of solutions of (0.4), which were called normal, was found by the author in [5]. He utilized the method of separation of variables explained in [6].* This system consists of the following vector-functions (e_r, e_θ and e_φ are unit vectors in spherical coordinates)

$$\mathbf{u}_{ln} = r^{-l}\left(\beta_l Y_{ln}\mathbf{e}_r + \frac{\delta_l}{\sqrt{l(l+1)}}\frac{\partial}{\partial\theta}Y_{ln}\mathbf{e}_\theta + \frac{\delta_l}{\sqrt{l(l+1)}}\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}Y_{ln}\mathbf{e}_\varphi\right)$$

$$\mathbf{v}_{ln} = r^{-(l+2)}\left(Y_{ln}\mathbf{e}_r - \frac{1}{l+1}\frac{\partial}{\partial\theta}Y_{ln}\mathbf{e}_\theta - \frac{1}{(l+1)\sin\theta}\frac{\partial}{\partial\varphi}Y_{ln}\mathbf{e}_\varphi\right)$$

$$\mathbf{w}_{ln} = \frac{i}{\sqrt{l(l+1)}}r^{-(l+1)}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}Y_{ln}\mathbf{e}_\theta - \frac{\partial}{\partial\theta}Y_{ln}\mathbf{e}_\varphi\right) \tag{1.1}$$

$$\mathbf{p}_{ln} = r^{l+1}\left(\alpha_l Y_{ln}\mathbf{e}_r - \frac{\gamma_l}{\sqrt{l(l+1)}}\frac{\partial}{\partial\theta}Y_{ln}\mathbf{e}_\theta - \frac{\gamma_l}{\sqrt{l(l+1)}}\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}Y_{ln}\mathbf{e}_\varphi\right)$$

$$\mathbf{q}_{ln} = r^{l-1}\left(Y_{ln}\mathbf{e}_r + \frac{1}{l}\frac{\partial}{\partial\theta}Y_{ln}\mathbf{e}_\theta + \frac{1}{l\sin\theta}\frac{\partial}{\partial\varphi}Y_{ln}\mathbf{e}_\varphi\right)$$

$$\mathbf{r}_{ln} = \frac{i}{\sqrt{l(l+1)}}r^{l}\left(\frac{1}{\sin\theta}Y_{ln}\mathbf{e}_\theta - \frac{\partial}{\partial\theta}Y_{ln}\mathbf{e}_\varphi\right)$$

where

$$Y_{ln}(\theta,\varphi) = P_{ln}(\cos\theta)e^{in\varphi} \quad (l=0,1,2,\ldots;\ -l\leqslant n\leqslant l)$$

P_{ln} is the conjugate Legendre function determined by the formula

$$P_{ln}(x) = \frac{(1-x^2)^{n/2}}{2^l l!}\frac{d^{l+n}}{dx^{l+n}}(x^2-1)^l$$

$$\alpha_l = \frac{2\rho-(1-\rho)l}{2l+3}, \qquad \gamma_l = \frac{l[(1-\rho)l+3-\rho]}{(2l+3)\sqrt{l(l+1)}}$$

$$\left(\rho = \frac{1-2\sigma}{2(1-\sigma)}\right) \tag{1.2}$$

$$\beta_l = \frac{(\rho-1)l-(\rho+1)}{2l-1}, \qquad \delta_l = \frac{l^2(1-\rho)-l(1+\rho)-2}{(2l-1)\sqrt{l(l+1)}}$$

The solutions $\mathbf{u}_{ln}$, $\mathbf{v}_{ln}$, $\mathbf{w}_{ln}$ are called outer normal solutions, and $\mathbf{p}_{ln}$, $\mathbf{q}_{ln}$ and $\mathbf{r}_{ln}$ inner normal solutions.

(2) We will find now transformation formulas, which will express inner normal vectors referred to the coordinate system with the origin at point O_{10} (Fig.1), by the inner normal vectors referred to the coordinate system *XYZ* with the origin at the point *O*. (The axes of both coordinate

* It could be also found more easily by a method of spherical vectors (see [7]).

systems are parallel).

We note, first of all, that

$$\operatorname{div} \mathbf{u}_{ln} = P_{ln}(\cos\theta)\, e^{in\varphi} / r^{l+1}, \quad \operatorname{div} \mathbf{p}_{ln} = P_{ln}(\cos\theta)\, e^{in\varphi} r^{l} \tag{1.3}$$

$$\operatorname{div} \mathbf{w}_{ln} = 0, \qquad \operatorname{div} \mathbf{r}_{ln} = 0 \tag{1.4}$$

$$\operatorname{div} \mathbf{v}_{ln} = 0, \qquad \operatorname{div} \mathbf{q}_{ln} = 0 \tag{1.5}$$

It is easy to verify that $\mathbf{v}_{ln}$ and $\mathbf{q}_{ln}$ are gradients of harmonic functions, namely

$$\begin{gathered} \mathbf{v}_{ln}(r, \theta, \varphi) = \operatorname{grad}\left(-\frac{1}{l+1}\frac{P_{ln}(\cos\theta)\, e^{in\varphi}}{r^{l+1}}\right) \\ \mathbf{q}_{ln}(r, \theta, \varphi) = \operatorname{grad}\left(\frac{1}{l} P_{ln}(\cos\theta)\, e^{in\varphi} r^{l}\right) \end{gathered} \tag{1.6}$$

(3) We seek in the transformation formulas for the $\mathbf{u}_{ln}$ in the form

$$\mathbf{u}_{ln}(r_{10}, \theta_{10}, \varphi) = \sum \alpha_{lkn} \mathbf{p}_{kn}(r, \theta, \varphi) + \sum \beta_{lkn} \mathbf{q}_{kn}(r, \theta, \varphi) + \sum \gamma_{lkn} \mathbf{r}_{kn}(r, \theta, \varphi) \tag{1.7}$$

where the summation limits are determinate simultaneously with the coefficients α_{lkn}, β_{lkn}, γ_{lkn}.

To find α_{lkn} we take the divergence of the left-and right-hand sides of (1.7). In view of (1.3), (1.4) and (1.5) and considering the transformation formula for the spherical functions (the method of obtaining such a formula is shown in [2], page 136)

$$\frac{P_{ln}(\cos\theta_{10})\, e^{in\varphi}}{r_{10}^{\,l+1}} = \sum_{k=n}^{\infty} \frac{(-1)^{l-n}}{d^{l+k+1}} \frac{(l+k)!}{(k+n)!\,(l-n)!} P_{kn}(\cos\theta)\, r^{k} e^{in\varphi} \quad (r < d) \tag{1.8}$$

we get

$$\alpha_{lkn} = \frac{(-1)^{l-n}}{d^{l+k+1}} \frac{(l+k)!}{(k+n)!\,(l-n)!} \qquad (k = n, n+1, \ldots) \tag{1.9}$$

To find γ_{lkn} it is convenient first to find the curl of the left- and right-hand side of (1.7), since α_{lkn} is already known. From (1.6) it follows that

$$\operatorname{curl} \mathbf{q}_{ln} = 0 \tag{1.10}$$

Finding by means of well-known formulas any component of the curl of the left- or right-hand side of (1.7), the ϕ-component,* say, and using

* The ϕ-components curl $\mathbf{u}_{ln}$, curl $\mathbf{p}_{ln}$, curl $\mathbf{r}_{ln}$, and curl $\mathbf{w}_{ln}$ are spherical functions (harmonics).

again (1.8), we get

$$\gamma_{lkn} = \frac{(-1)^{l-n+1}}{d^{l+k}} \frac{2n}{lk} \sqrt{\frac{k}{k+1}} \frac{(l+k)!}{(k+n)!\,(l-n)!} \qquad (k = n, n+1, \ldots) \qquad (1.11)$$

To find β_{lkn} we write the vector relationship (1.7) in terms of the ϕ-components. Taking into account the expressions for a_{lkn} and γ_{lkn}, and the transformation formula (1.8), we have

$$\beta_{lkn} = \frac{(-1)^{l-n+1}}{d^{l+k-1}} \frac{(l+k-1)!}{(k+n)!\,(l-n)!} (\sigma_{k-1,l} + 2n^2\tau_{k-1,l}) \qquad (1.12)$$

where

$$\sigma_{kl} = \frac{8(\rho-1)k^2l^2 - 2(\rho-1)(l^2+k^2) + (\rho+3)(l-2kl-k)}{(4k^2-1)(4l^2-1)}$$

$$\tau_{kl} = \frac{4(\rho-5)k^2l^2 + 8(l^2+k^2) + (\rho+3)lk(2k-2l-lk)}{kl(4k^2-1)(4l^2-1)} \qquad (1.13)$$

(4) The transformation formula for $\mathbf{v}_{ln}$ is immediately found from (1.6) and (1.8)

$$\mathbf{v}_{ln}(r_{10}, \theta_{10}, \varphi) = \sum_{k=n}^{\infty} \frac{(-1)^{l-n+1}}{d^{l+k+1}} \frac{k(l+k)!}{(l+1)(k+n)!\,(l-n)!} \mathbf{q}_{ln}(r, \theta, \varphi) \qquad (r < d) \qquad (1.14)$$

(5) We shall seek the transformation formula for $\mathbf{w}_{ln}$ in the form analogous to (1.7). Since div $\mathbf{w}_{ln}$ = div $\mathbf{r}_{ln}$ = div $\mathbf{q}_{ln}$ = 0 the coefficients of $\mathbf{p}_{kn}$ are zero and the formula has the following form:

$$\mathbf{w}_{ln}(r_{10}, \theta_{10}, \varphi) = \sum \lambda_{lkn}\mathbf{r}_{kn}(r, \theta, \varphi) + \sum \mu_{lkn}\mathbf{q}_{kn} \qquad (r, \theta, \varphi) \qquad (1.15)$$

Prior to finding λ_{lkn} and μ_{lkn}, we notice that

$$\operatorname{div}\operatorname{curl}\mathbf{w}_{ln} = 0, \qquad \operatorname{div}\operatorname{curl}\mathbf{r}_{ln} = 0$$

Moreover, from (0.4) we get

$$\Delta\mathbf{u} = \operatorname{grad}\operatorname{div}\mathbf{u} - \operatorname{curl}\operatorname{curl}\mathbf{u}$$

and it follows from (1.5) that

$$\operatorname{curl}\operatorname{curl}\mathbf{w}_{ln} = 0, \qquad \operatorname{curl}\operatorname{curl}\mathbf{r}_{ln} = 0$$

i.e.

$$\operatorname{curl}\mathbf{w}_{ln} = \operatorname{grad}\varphi_{ln}, \qquad \operatorname{curl}\mathbf{r}_{ln} = \operatorname{grad}\psi_{ln}$$

where ϕ_{ln} and ψ_{ln} are harmonic functions.

Finding the ϕ-component of the curl of the left- and right-hand side of (1.15), we get

$$\varphi_{ln} = i\sqrt{\frac{l}{l+1}} \frac{P_{ln}e^{in\varphi}}{r^{l+1}}, \qquad \psi_{ln} = -i\sqrt{\frac{l+1}{l}} P_{ln}r^l e^{in\varphi}$$

and, taking into consideration (1.10) and (1.8) we get

$$\lambda_{lkn} = \frac{(-1)^{l-n+1}}{d^{l+k+1}} \sqrt{\frac{lk}{(l+1)(k+1)} \frac{(l+k)!}{(k+n)!\,(k-n)!}} \qquad (k = n,\ n+1, \ldots) \tag{1.16}$$

Now, writing (1.15) for the ϕ-component vectors and substituting values of λ_{lkn}, we find μ_{lkn} :

$$\mu_{lkn} = \frac{(-1)^{l-n}\, n}{d^{l+k}\sqrt{l(l+1)}} \frac{(l+k)!}{(k+n)!\,(l-n)!} \qquad (k = n+1,\ k+2, \ldots) \tag{1.17}$$

(6) Thus, the transformation formulas are obtained.* It is easy to verify using D'Alembert's criterion and the estimate

$$|P_{ln}(\cos\theta)| < \sqrt{\frac{(l+n)!}{(l-n)!}} \tag{1.18}$$

that the series which appear in the right-hand sides of these formulas are uniformly convergent in an arbitrary sphere with radius smaller than d.

The validity of the found vector relations can be easily verified by explicitly writing these vectors in all three Cartesian components.**

For convenience, the transformation formulas for the normal solutions are written in the following form (here, however, the same letters denote different coefficients)

$$\mathbf{u}_{ln}(r_{10}, \theta_{10}, \varphi) = \sum_{k=n}^{\infty} \frac{(-1)^{l-n}}{d^{l+k+1}} \alpha_{lkn}\mathbf{p}_{kn}(r, \theta, \varphi) +$$

$$+ \sum_{k=n+1}^{\infty} \frac{(-1)^{l-n+1}}{d^{l+k-1}} \beta_{lkn}\mathbf{q}_{kn}(r, \theta, \varphi) + \sum_{k=n}^{\infty} \frac{(-1)^{l-n+1}}{d^{l+k}} \gamma_{lkn}\mathbf{r}_{kn}(r, \theta, \varphi) \tag{1.19}$$

$$\mathbf{v}_{ln}(r_{10}, \theta_{10}, \varphi) = \sum_{k=n}^{\infty} \frac{(-1)^{l-n+1}}{d^{l+k+1}} \delta_{lkn}\,\mathbf{q}_{kn}(r, \theta, \varphi)$$

* In an analogous manner we could find transformation formulas for $r > d$, but we shall not need them in the sequel.

** In the right-hand and left-hand sides of the resulting equalities we obtain biharmonic functions. The transformation formulas for such functions are easily found from (1.18).

$$\mathbf{w}_{ln}(r_{10}, \theta_{10}, \varphi) = \sum_{k=n+1}^{\infty} \frac{(-1)^{l-n}}{d^{l+k}} \mu_{lkn} \mathbf{q}_{kn}(r, \theta, \varphi) +$$

$$+ \sum_{k=n}^{\infty} \frac{(-1)^{l-n+1}}{d^{l+k+1}} \lambda_{lkn} \mathbf{r}_{kn}(r, \theta, \varphi)$$

where (1.20)

$$\alpha_{lkn} = \frac{(l+k)!}{(k+n)!\,(l-n)!}, \qquad \beta_{lkn} = \frac{(l+k-1)!}{(k+n)!\,(l-n)!}(\sigma_{k-1,\,l} + 2n^2\tau_{k-1,\,l})$$

$$\gamma_{lkn} = \frac{2n}{l\sqrt{k(k+1)}} \frac{(l+k)!}{(k+n)!\,(l-n)!}, \qquad \delta_{lkn} = \frac{k}{l+1} \frac{(l+k)!}{(k+n)!\,(l-n)!}$$

$$\lambda_{lkn} = \sqrt{\frac{lk}{(l+1)(k+1)}} \frac{(l+k)!}{(k+n)!\,(l-n)!}, \qquad \mu_{lkn} = \frac{n}{\sqrt{l(l+1)}} \frac{(l+k)!}{(k+n)!\,(l-n)!}$$

and σ_{kl} and τ_{kl} are found from (1.13).

Note. We obtained the transformation formulas so to speak for "above-down" (Fig.1). The formulas of transformation for "below-up", i.e. formulas expressing, for instance $\mathbf{u}_{ln}(r, \theta, \phi)$ $\mathbf{p}_{ln}(r_{10}, \theta_{10}, \phi)$, etc., are easily found from already obtained formulas. Indeed, reversing the direction of the z-axis, we express using (1.8) $P_{ln}(\cos\psi)\, e^{in\phi}/r^{l+1}$ by $P_{kn}(\cos\psi_1)\, r_{10}{}^k e^{in\phi}$ (Fig.1), then we take into account that $\psi = \pi - \theta$, $\psi_1 = \pi - \theta_{10}$ and

$$P_{ln}(-x) = (-1)^{l-n} P_{ln}(x) \tag{1.21}$$

2. Formulation and solution of problems.

(1) Using a system of normal solutions and the transformation formulas for these systems, we are now in a position to construct solutions for a layer with a spherical cavity and with the following boundary conditions on the outer faces A and B (Fig.1).*

$$\sigma_{zx} = \sigma_{zy} = 0, \qquad u_z|_A = -c, \qquad u_z|_B = c, \qquad c = \text{const} \tag{2.1}$$

These conditions are equivalent to the uniform compression of a layer by a rigid punch without friction between the punch and the layer. On the boundary of the cavity we may assign various conditions of the form:

$$\sum_{i=1}^{3} a_{ij} u_i + \sum_{i=1}^{3}\sum_{k=1}^{3} b_{ikj}\sigma_{ik} = f_j(\theta, \varphi) \qquad (j = 1, 2, 3,\ b_{ikj} = b_{kij}) \tag{2.2}$$

* σ_{zx}, $\sigma_{r\theta}$, etc., are components of the stress tensor in corresponding coordinate systems.

where a_{ij} and b_{ikj} are constants; i, k = 1, 2, 3, thus, the numbers 1, 2, 3 correspond to the subscripts r, θ and ϕ; in other words, the components of the displacement vector and of the stress tensors are expressed in spherical coordinates. The three equalities (2.2), which correspond to the three different values of j, are linearly independent.

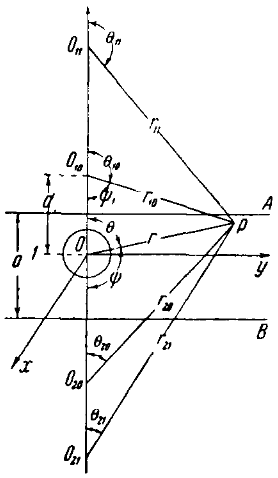

Fig. 1.

It is required of the functions $f_j(\theta, \phi)$, (j = 1, 2, 3) that they can be expanded into a uniformly convergent $Y_{ln}(\theta, \phi)$ - series; such that their Fourier coefficients should satisfy the following inequalities

$$\lim_{k\to\infty} f_{kj} \mid f_{k-1,\, j} = p_j < 1 \qquad (j = 1, 2, 3)$$

(These relationships represent sufficient conditions to prove convergence of the series representing a solution. They are not, however, with all probability, necessary conditions.)

Using the same scheme as above, problems with different types of boundary conditions on A and B, can be solved, viz. when

$$u_x = u_y = 0, \qquad \sigma_{zz} = p = \text{const} \tag{2.3}$$

The solutions of the boundary value problems are sought in the region 1 (Fig. 1). Besides, the following conditions at infinity have to be satisfied [8]:

$$r(\mathbf{u} - \mathbf{u}_0) = 0(1), \qquad r^2(T\mathbf{u} - T\mathbf{u}_0) = 0(1) \tag{2.4}$$

where r is the radius vector, measured from an arbitrary point; T is an operator of the stresses, such that

$$(T\mathbf{u})_i = \sigma_{ik} n_k$$

(n_k are directional cosines of the normal at a given point to the surface; i, k = 1, 2, 3 which corresponds to $x_1 = x$, $x_2 = y$, $x_3 = z$); vector $\mathbf{u}_0$ is the same u_0 which was mentioned in the introduction when the method applied here was explained, i.e. it is a solution of the boundary value problem with the same conditions on the faces of the layer, but without the cavity.

It is easy to verify that for the problems with the boundary conditions (2.1) the following is true:

$$u_{0z} = \frac{2c}{a} z = \frac{2c}{a} r P_1(\cos\theta), \qquad u_{0x} = u_{0y} = 0 \tag{2.5}$$

and for the problems with the boundary conditions (2.3), we have

$$u_{0z} = -\frac{(1+\sigma)(1-2\sigma)}{E(1-\sigma)} pz, \qquad u_{0x} = u_{0y} = 0$$ (E is the modulus of elasticity)

Solutions of any of the above mentioned boundary value problems are constructed in vector form according to the scheme explained in the introduction. Here, however, spherical functions are replaced by the outer normal solutions.

(2) Let us construct, for example, a solution of the boundary value problem: $u_r = u_\theta = u_\phi = 0$, for $r = R$ (this corresponds to a rigid spherical inclusion with an infinite friction), and with the boundary conditions (2.1) on the faces of the layer. We seek a solution $\mathbf{u}$ in the form

$$\mathbf{u} = \mathbf{u}_0 + \mathbf{u}_1 \tag{2.6}$$

where $\mathbf{u}_0$ is determinate by the formula (2.5), and $\mathbf{u}_1$ is constructed in the form of the series obtained from the axisymmetrical solutions $\mathbf{u}_{l0}$ and $\mathbf{v}_{l0}$, ($\mathbf{w}_{l0} \equiv 0$):

$$\mathbf{u}_1 = \sum_{l=1}^{\infty} A_l R^{l+1} \mathbf{u}_{l0}(r, \theta) + \sum_{l=0}^{\infty} B_l R^{l+3} \mathbf{v}_{l0}(r, \theta) +$$
$$+ \sum_{i=1}^{2} \sum_{\mu=0}^{\infty} \sum_{l=1}^{\infty} A_{l\mu} R^{l+1} \mathbf{u}_{l0}(r_{i\mu}, \theta_{i\mu}) + \sum_{i=1}^{2} \sum_{\mu=0}^{\infty} \sum_{l=0}^{\infty} B_{l\mu} R^{l+3} \mathbf{v}_{l0}(r_{i\mu}, \theta_{i\mu}) \tag{2.7}$$

where A_l, $A_{l\mu}$, B_l and $B_{l\mu}$ are unknown coefficients, r and $r_{i\mu}$, θ and $\theta_{i\mu}$ are the distances and angles respectively, measured as it is shown in Fig. 1, (the points $O_{i\mu}$ (i = 1, 2; μ = 0, 1, 2, ...) are mirror images of the center of the cavity, O, in the face-planes of the layer).

The vector $\mathbf{u}_1$ has clearly to satisfy the homogeneous boundary conditions on the planes A and B

$$u_{1z} = 0, \quad \sigma_{zx} = \sigma_{zy} = 0 \tag{2.8}$$

Using recurrence formulas for the functions P_{ln} and the formulas for differentiation of the spherical functions [2], we find, after some elementary calculations, that $(\mathbf{u}_{l0})_z$ and $(\mathbf{v}_{l0})_z$ and the components of the stress tensors corresponding to (2.8) contain linear combinations of the functions $P_{l\pm1}/r^l$, P_{l+1}/r^{l+2}, $P_{l,1}/r^{l+1}$, $P_{l+2,1}/r^{l+1}$, $P_{l+2,1}/r^{l+3}$.

We find, for instance, as a consequence of this, and in view of the fact that on the plane A (Fig. 1),

$$r = r_{10}, \qquad r_{2\mu} = r_{1,\mu+1}, \qquad \cos\theta_{10} = -\cos\theta, \qquad \cos\theta_{2\mu} = -\cos\theta_{1,\mu+1}$$

and taking into consideration (1.21) that to satisfy conditions (2.8) on the planes A and B the following relationships between the coefficients of the series are sufficient

$$A_{l\mu} = (-1)^{l(\mu+1)} A_l, \qquad B_{l\mu} = (-1)^{l(\mu+1)} B_l \tag{2.9}$$

To fulfill the boundary conditions on the surface of the cavity, we first reduce $\mathbf{u}_1$ to a single set of independent variables r, θ and ϕ.

Considering that the center of the cavity is equidistant from the upper and lower faces of the layer.* Using transformation formulas (1.19), (see also remark at the end of Section 1), and interchanging the order of summation, we get

$$\begin{aligned}
\mathbf{u}(r,\theta) &= \sum_{k=1}^{\infty} A_k R^{k+1}\mathbf{u}_{k0}(r,\theta) + \sum_{k=0}^{\infty} B_k R^{k+3}\mathbf{v}_{k0}(r,\theta) + \\
&+ \sum_{k=0}^{\infty} \frac{\mathbf{p}_{k0}(r,\theta)}{a^k} \sum_{l=1}^{\infty} A_l[(-1)^l + (-1)^k]\,\alpha_{lk0}\left(\frac{R}{a}\right)^{l+1} \eta(l+k+1) - \\
&- \sum_{k=0}^{\infty} \frac{\mathbf{q}_{k0}(r,\theta)}{a^{k-2}} \sum_{l=1}^{\infty} A_l[(-1)^l + (-1)^k]\,\beta_{lk0}\left(\frac{R}{a}\right)^{l+1} \eta(l+k-1) - \\
&- \sum_{k=0}^{\infty} \frac{q_{k0}(r,\theta)}{a^k} \sum_{l=0}^{\infty} B_l[(-1)^l + (-1)^k]\,\delta_{lk0}\left(\frac{R}{a}\right)^{l+1} \eta(l+k-1)
\end{aligned} \tag{2.10}$$

$$\eta(p) = \sum_{\mu=0}^{\infty} \frac{(-1)^{l(\mu+1)}}{(\mu+1)^p}$$

* In case of an arbitrary location of the center of the cavity, expressions (2.10), and consequently the system (2.11), are more cumbersome.

where α_{lk0}, β_{lk0} and δ_{lk0} are determined by (1.20).

Now, from the boundary conditions $u_r = 0$ and $u_\theta = 0$, for $r = R$, ($u\phi = 0$ since $(\mathbf{u}_{l0})_\phi \equiv (\mathbf{v}_{l0})_\phi \equiv 0$), we find the following system of relations for determination of the coefficients A_l and B_l (here $u = R/A$)

$$B_0 + \sum_{l=2}^{\infty} \frac{4}{3} A_l u^{l+1} \eta(l+1) = -\frac{c}{3a} \qquad (l \text{ is even}) \tag{2.11.1}$$

$$A_k\beta_k + B_k + \alpha_k \sum_{l=1}^{\infty} A_l [(-1)^l + (-1)^k] \alpha_{lk0} u^{l+k+1} \eta(l+k+1) -$$
$$- \sum_{l=1}^{\infty} A_l [(-1)^l + (-1)^k] \beta_{lk0} u^{l+k-1} \eta(l+k-1) -$$
$$- \sum_{l=0}^{\infty} B_l [(-1)^l + (-1)^k] \delta_{lk0} u^{l+k+1} \eta(l+k+1) = -\frac{4c}{3a} \delta_{k2}$$

$$(k = 1, 2, 3, \ldots; \; l = 0, 1, 2, \ldots; \; \delta_{km} = 0 \text{ for } k \neq m, \; \delta_{mm} = 1) \tag{2.11.2}$$

$$A_k \frac{\delta_k}{\sqrt{k(k+1)}} - B_k \frac{1}{k+1} - \frac{\gamma_k}{\sqrt{k(k+1)}} \sum_{l=1}^{\infty} A_l [(-1)^l +$$
$$+ (-1)^k] \alpha_{lk0} u^{l+k+1} \eta(l+k+1) - \frac{1}{k} \sum_{l=1}^{\infty} A_l [(-1)^l +$$
$$+ (-1)^k] \beta_{lk0} u^{l+k-1} \eta(l+k-1) - \frac{1}{k} \sum_{l=0}^{\infty} B_l [(-1)^l +$$
$$+ (-1)^k] \delta_{lk0} \eta(l+k+1) = -(2c/3a) \delta_{k2} \qquad (k = 1, 2, 3, \ldots; \; l = 0, 1, 2) \tag{2.11.3}$$

Subtracting equations (2.11.3) from the equation (2.11.2) divided by k, and introducing new coefficients

$$x_k = A_k\beta_k + B_k$$

$$y_k = A_k \left(\frac{\beta_k}{k} - \frac{\delta_k}{\sqrt{k(k+1)}} \right) + B_k \frac{2k+1}{k(k+1)} \equiv \frac{\rho - 1}{k} A_k + \frac{2k+1}{k(k+1)} B_k \tag{2.12}$$

and then putting

$$x_k = z_k, \qquad y_k = z_{k+1}, \qquad k = 1, 2, 3, \ldots \tag{2.13}$$

we can write the system of equations (2.11) in the form of (0.2).

(3) Now we shall show that the inequalities (0.3) are true. To do this we notice that

$$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} D_{kl}^2 < \infty \qquad \left(D_{kl} = \frac{(k+l)!}{k!\,l!}u^{k+l}, \quad u < \frac{1}{2}\right) \tag{2.14}$$

Indeed, it is true that

$$\sum_{l=1}^{\infty} D_{kl} = \frac{u^k}{k!}\sum_{l=1}^{\infty}\frac{\partial^k u^{k+l}}{\partial u^k} = \left(\frac{u}{1-u}\right)^k \tag{2.15}$$

from which (2.14) follows.

The fact that C_{kl}, (the coefficients for z_e in the transformed system (2.11)), are different from D_{kl}, does not prevent the satisfaction of the first inquality in (0.3). The second inequality is verified immediately.

Thus, the matrix formed by the coefficients C_{kl} is a uniformly continuous operator in Hilbert space l_2. Moreover, the free term b_k also belongs to the same space.

Consequently, for the system (2.11) the Fredholm's collorary is true. However, the corresponding homogeneous system obtained for $c = 0$, i.e. for the zero boundary conditions, (see (2.1)), cannot have a nontrival solution. This follows from the uniqueness theorem for the boundary value problems formulated here.

It follows then, that the system (2.11) has a single bounded solution for arbitrary right-hand expressions.

The proof that the series obtained above are convergent and together with $\mathbf{u}_0$ yield solution of the problem, and the proof that, in particular, we can differentiate the series (2.7) term by term, is conducted by substituting the coefficients A_l and B_l (all other are expressible by them) by their majors A_l' and B_l', for which

$$\lim_{l\to\infty}(A_l'/A_{l-1}') < 1, \qquad \lim_{l\to\infty}(B_l'/B_{l-1}') < 1 \tag{2.16}$$

The behavior of the normal solutions is evaluated with the help of the inequality (1.18).

The possibility of introducing the coefficients A_l' and B_l' is seen from the following.

If in the system (2.11), written in the form (0.2), C_{kl} are replaced by the coefficients D_{kl}, then, taking into account (2.15) and the boundedness of the solution of the system, we obtain the following estimate

$$|z_k| \leqslant z_k' = z_{\max}\left(\frac{u}{1-u}\right)^k + |b_k|$$

From here

$$\lim_{k\to\infty}(x_k'/x'_{k-1}) = \lim_{k\to\infty}(y_k'/y'_{k-1}) < 1 \tag{2.17}$$

Replacing in (2.12) x_k and y_k by x_k' and y_k' we obtain A_k' and B_k for which (2.16) is true. In a similar manner, but slightly more cumbersome, one finds the coefficients A_k' and B_k' if one does not substitute in the system (2.11) C_{kl} by D_{kl}.

(4) In an analogous manner, using axisymmetrical vectors $\mathbf{u}_{l0}$ and $\mathbf{v}_{l0}$, the boundary value problems with the following boundary conditions on the surface of the cavity are solved:

(i) $\sigma_{rr} = \sigma_{r\theta} = \sigma_{r\phi} = 0$ (free surface)

(ii) $u_r = 0$; $\sigma_{r\theta} = \sigma_{r\phi} = 0$ (rigid spherical inclusion without friction), and other axisymmetrical problems.

The speed of convergence of the series can be judged from the results obtained in the problems related to electrostatics, (see [1]). In that case it was shown that in some special instances the series representing the coefficients of the field intensity can be replaced by the first two or three terms. The error thus evolved was only about 1 per cent.

In the case of n spherical cavities with the centers on one straight line perpendicular to the face-planes of the layer, the solution $\mathbf{u}_1$ is sought in the form of the series related to the centers of all cavities and their mirror images in the planes A and B.

Consequently, instead of the two series of unknown coefficients A_l and B_l, $\mathbf{u}_1$ will be composed of n such series. Moreover, for the determination of these new coefficients, one obtains a system of a similar form as considered above.

In the case of non-axisymmetrical problems $\mathbf{u}_1$ is sought in the form not only in terms of $\mathbf{u}_{l0}$ and $\mathbf{v}_{l0}$ but also in terms of all normal solutions. (If the spherical cavities of a layer are filled with a substance having different elastic characteristic σ and E than the layer, then, clearly, the solution inside of the cavities should be sought in the form of the series in terms of the inner normal vectors).

(5) To solve problems with more general conditions on the face planes, when the right-hand sides of the expressions (2.1) and (2.3) are functions of the points of the plane, it is possible to find Green's tensors using the above method, (see [5]). In addition to the transformation formulas, composition theorems for normal solutions, or rotation formulas which

express the normal vector $\mathbf{Q}_{ln}(r, \theta, \phi)$ in a given coordinate system by the normal vectors $\mathbf{Q}_{ln}(r, \theta', \phi')$, are used. The latter vectors are related to the coordinate system which is obtained from a given system through some rotation specified by Eulerian angles ϕ_1, a, ϕ_2. These formulas have the form:

$$\mathbf{Q}_{ln}(r, \theta, \varphi) = \sum_{s=-l}^{l} \mathbf{Q}_{ls}(r', \theta', \varphi') \sqrt{\frac{(l+n)!\,(l-s)!}{(l-n)!\,(l+s)!}}\, T_{sn}^{\;l}(\pi - \varphi_2, \alpha, \pi - \varphi_1) \tag{2.18}$$

and they are obtained using the addition theorem for the generalized spherical functions $T_{sn}^{\;l}(\phi_1, a, \phi_2)$ derived in [6].

Note. An attempt could be made to solve these problems using the above derived method, (without Green's tensors). In this case $\mathbf{u}_0$ could also be found (see [9]). However, the expansion of $\mathbf{u}_0$ on the surface of the cavity in terms of Y_{ln} presents in itself quite a complicated problem.

Notice that this method cannot be applied to the problems with different boundary conditions than those enumerated above. The problem of a semi-infinite layer with an arbitrary number of spherical cavities and with arbitrary centers can be solved with the help of rotation and translation formulas. Using the same method, the following problems can be solved: an infinite body with three spherical cavities (the problem of two spherical cavities can be solved using transformation formulas only) and the problem of a non-concentric cavity in a sphere.

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